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THE PROBLEM OF THE POLAR PLANIMETER.

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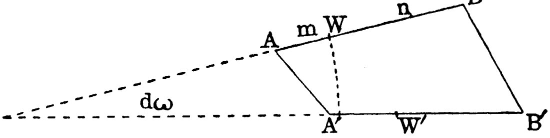
I have found it interesting to consider the general problem of the planimeter as that of finding the area swept over by a right line of constant length moving in a plane, and having an attachment for recording the normal movement of a point on it. It appears to be more simple than the demonstrations usually given in the text books for the action of the instrument in special cases.



Let AB be the line of constant length l, and let the wheel be on the line at a distance m from A, and n from B, and at a distance c from the middle of AB.

Let AB make a small shift to $A'B'$, turning through the angle $d\omega$. Let ρ be the distance of A from the point of contact of the line with its envelope.

The wheel record is

$$dW = (\rho + m)d\omega;$$

whence

$$l\rho d\omega = ldW - lmd\omega.$$

The elementary area swept over by the line is

$$\begin{aligned} d\Omega &= \tfrac{1}{2}(\rho + l)^2 d\omega - \tfrac{1}{2}\rho^2 d\omega, \\ &= ldW + \tfrac{1}{2}l(l - 2m)d\omega, \\ &= ldW + \tfrac{1}{2}(n^2 - m^2)d\omega, \\ &= ldW + cld\omega. \end{aligned}$$

$$\Omega = lW + cl.\int d\omega$$

is the general equation of the planimeter, giving the area between the curves (A) and (B) described by the extremities of l, and the initial and final positions of l.

All particular cases which can exist are determined by the limits of $\int d\omega$ and the direction in which the line turns. If we consider the wheel record to increase and the line to move through positive angles as it turns around A as a centre in the direction of clock hands, then for this motion both dW and $d\omega$ are positive; for the opposite motion about A they are both negative. If l turns about B clockwise, dW is negative and $d\omega$ positive, and their signs are changed when the direction of motion is reversed.

If the extremities describe closed curves of areas (A) and (B), and the area of one is known, that of the other is determined.

1. If the line does not revolve, but oscillates back to its first position, then $\int d\omega = 0$ always, and

$$\Omega = (A) \sim (B) = lW.$$

If A or B moves in a line of any kind and retraces its path while the other describes a closed curve, then the area of that curve is lW.

In particular, if the line traveled by the oscillating point be straight or circular, we have the rolling planimeter whose roller is a cylinder or frustum of a cone of revolution.

It is therefore not necessary, so far as accuracy is concerned, that the two rollers should be segments of the same cylinder in the construction of the actual instrument.

2. If the line does revolve once then the formula for the area between the curves is

$$\Omega = (A) \sim (B) = lW + 2\pi cl,$$

in which the signs of the two terms are determined as above. The sign of W, however, always takes care of itself, as shown by the wheel readings.

If the curve (B) envelops the curve (A) the signs of both terms are the same; and if (A) envelops (B) they have different signs for position of roller nearer A than B.

If the path of one of the points crosses that of the other, then the formula for the area between the curves gives the difference between the sum of the areas which lie on one side of one of the curves and the sum of those which lie on the other side of it. In which case if the area of this curve be added (with proper sign) to this difference the area of the other closed curve is determined.

The area of one of the curves may be zero, in which case the area of the other is given, and we may have a constant to add for the rolling planimeter; but here nothing is gained by complete revolution, and the mechanism of the instrument prevents it.

In particular, if the point A moves on the circumference of a circle of radius r,

$$\Omega = lW + \pi(n^2 - m^2),$$

$$(B) = lW + \pi(n^2 - m^2) + (A),$$

$$= lW + \pi(l^2 + r^2 + 2ml),$$

and we have the polar planimeter as usually constructed where $n = l + m$, the second term being *the constant* for the instrument and the first the registration of the roller.

If $c = 0$, or the wheel is in the middle of the bar, then

$$d\Omega = l\,dW,$$

and the wheel record is the area between the curves always. Which agrees with the Properties of Guldinus.

If the wheel be attached to the bar by an offset of length b which is distant m and n from A and B respectively, the formulæ are identical with those above; for if the shift of AB is small and finite the wheel record is

$$\Delta W = (\rho + m)\Delta\omega + b(1 - \cos\Delta\omega),$$

the last term of which disappears in the limit. The normal movements of all points in the same perpendicular to a line are the same. Hence, if the plane of the wheel be normal to the line at its middle point, the area swept over is always lW, the wheel record, and no constant is to be added for revolution.

The formula for the area swept over by a right line moving in a plane in terms of the normal movement of a point on it furnishes an elegant proof of Holditch's Theorem.

Let P be a point on the line AB, of variable length λ, dividing AB in the constant ratio a/b.

If dW be the normal movement of the middle point C of AB, the elementary area swept over by AB is

$$d\Omega = \lambda\,dW.$$

That swept over by AP is

$$d\Omega' = \frac{a}{a+b}\lambda\,dW + \tfrac{1}{2}(\overline{PC}^2 - \overline{AC}^2)\,d\omega,$$

$$= \frac{a}{a+b}\,d\Omega - \frac{ab}{(a+b)^2}\cdot\tfrac{1}{2}\lambda^2 d\omega.$$

Integrating for a complete circuit and writing (O) for the area of the closed curve $\lambda = f(\omega)$ we have

$$(P) - (A) = \frac{a}{a+b}[(B) - (A)] - \frac{ab}{(a+b)^2}(O),$$

or

$$(P) = \frac{a(B) + b(A)}{a+b} - \frac{ab}{(a+b)^2}(O).$$